Math Review

Summer 2017

*Topic 3*

3. Calculus, differentiation and integrals

3.1. One-variable calculus and rules of differentiation

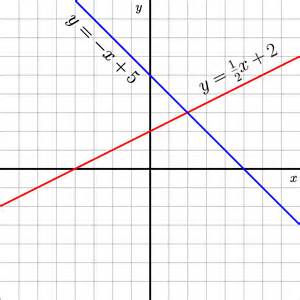
Calculus is a key tool for the micro series. We review the key definitions and work through a bunch of examples and exercises diligently. One gets a lot more by doing rather than reading through solutions. As you progress through your micro courses, you will likely forget some of the rules here and there, don’t feel discouraged, just look at some reference material and you will be fine.

3.1.1. Function

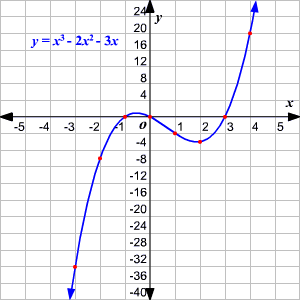
Let’s define a function in a one-variable calculus context.

Function: A function, takes and produces . A function may assign multiple number to one number, i.e.,, but in *one-variable calculus* a function simply maps one number to another number, i.e.,. If maps distinct points to distinct values, then f is *one-to-one* (or injective). Some examples of functions are:

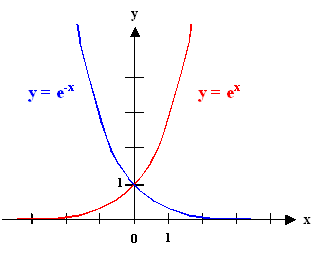
Linear function: , where is the slope and is the intercept



Polynomial function:

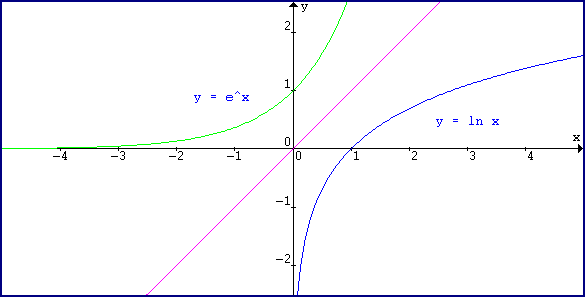


Exponential function:



Logarithmic function:

Key one is that of the :



There are other types of functions, but they are rarely used (at least not in your first year study) – for example trigonometric functions of the form

Here would be the *dependent variable*, also referred to as the *endogenous* variable and the *independent* or *exogenous* variable.

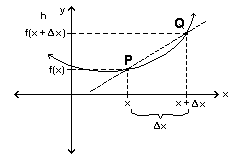
3.1.2. Derivatives

You will often hear about rate of change or the marginal effect. For example, when we have a production function for a good, we are usually interested in how a change in the inputs, , will affect the production of the good . The *derivative* of a function contains this information.

We will not go over too many details on derivatives as a function of limit, but it may helpful to recall that a derivative can be expressed as a limit.

The derivative of a function at is given by:

You will be able to take derivatives directly given the rules that you are familiar with (and that we will review today) but it is helpful to see an example of what is at work in the background. Taking derivatives are so easy that we rarely think back to where the basic formulas and rules originated.



*Example*. Use the limit definition to compute the derivative, for:

Can you try with

:

The function is differentiable at if and only if the limit exists. The function

is differentiable if and only if it is differentiable at every point .

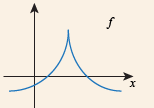
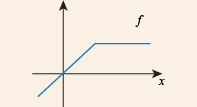
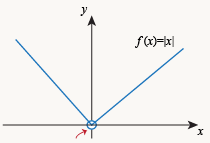
Let’s quickly recall our definition of continuous functions from *§ 2.7:*

Continuous function: A function is continuous at a point at if and only if, for every , there exists , such that:

If is differentiable at then is continuous at . So the differentiability of at is a sufficient condition to show that is continuous at . By the contrapositive we also have: If is not continuous at then is not differentiable at .

Q: Can a function be continuous at but not differentiable at ?

A: Yes, kinks and sharp ends.

Thus, we can say that a function is differentiable if it is both continuous and "smooth".

The derivative is the rate of change in and can be expressed as:

* If for , then is increasing on
* If for , then is decreasing on
* If for all , then we say that is *monotonically increasing* (or non-decreasing). If for all , then is *strictly increasing*.
* The point is a critical point of the continuous and differentiable function if or

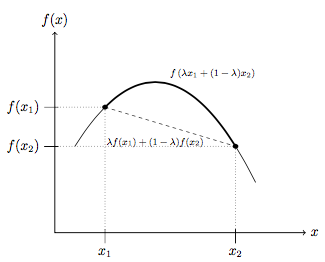
3.1.2 Second derivative

If the (first) derivative is a differentiable function, we can take its derivative, and get the *second derivative* which is expressed as:

* All higher order derivatives are defined in the same way. We also say that a function is *continuously differentiable* if it is *continuous, differentiable*, and the derivative is a *continuous function*.
* If a function possesses continuous derivatives it is called -times continuosly differentiable or

3.1.3. Derivatives, concavity and convexity of functions

Since we are on second derivatives, these are some useful results that you will encounter quite a bit in Micro.



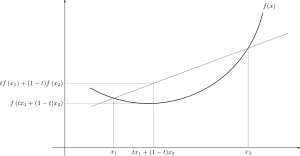
The function is concave if and only if for all . As you can see in the graph, if for any two points and any we have

, then is concave.

: How would you define a convex function? Draw a diagram to follow the logic.

: The function is convex if and only if for all . As in the graph, if for any two points and any we have

, then is convex.



If for all , then is strictly convex. If for all , then is strictly concave.

3.1.4 Rules of differentiation and practice

Some basic rules of differentiation are as follows:

1. For constant : = 0
2. For sums:
3. Power rule:
4. Product rule:
5. Quotient rule:
6. Chain rule:

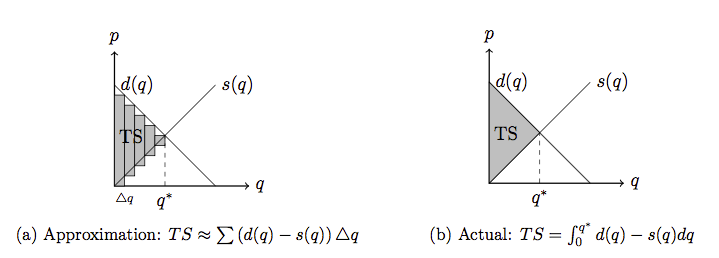
Let’s practice finding the derivatives of the following:

Professor Glewwe and Coggins have many instances where they extensively use derivatives with logs. Remember that if or if

Find the derivative of the following:

3.2 Integrals and related properties

In economics we are often interested in the integral of a function. Recall back to your basic demand and supply curves. The area under the curves gives us the total surplus. This area underneath the curves can be obtained using the actual integral or sometimes using approximations (approximate an integral by adding up the area of rectangles). You will see both in micro.



If we are *approximating* using the area of the rectangles, we are really finding the integral through the following:

The *actual* integral of a function is the limit of the above function as approaches zero. We have:

Integral. The integral of a function on is given by:

The function is integrable on if and only if the limit exists. We can also call the integral the *antiderivative* of If is continuous on then is also integrable on

*Properties*:

Suppose that and are integrable functions. Let be arbitrary constants. Then:



*The Fundamental Theorem of Calculus Theorem*. Let be a continuous function on the

open interval . If then:

We also have the following three properties:

1. If and then

Remember that if the integration does not have a specified range, then we need to include a constant of integration to the solution:

, where C is the constant of integration.

Remember that

: What is the

:

Let’s work through a simple example to refresh our memory of integrals.

*Example*. Calculate the integral of where .

Given that , we have

We can write

3.2.1. Integration by parts

Integration by parts is used a little less often but you may see it sometimes. It corresponds to the Product Rule for differentiation.

Others may remember it as letting and , then by substitution, we have:

*Example*. Find . Some may know this final result, but let’s try it using integration by parts.

There is not much you can pick for and , so this is pretty straightforward.

Let

Using integration by parts:

Q: Let’s have you try a slightly more involved example. Find: . *Hint, recall sometimes you have to repeat an integration by parts.*

Let

Let

Let

Let

3.2.2. Leibniz’s Rule

Finally, it might be helpful to reference the Leibniz’s Rule of differentiating integrals. It is a little complex looking, but an example may help.

Leibniz’s Rule:

*Example*. Calculate

Let’s think of the equivalence from the Rule,

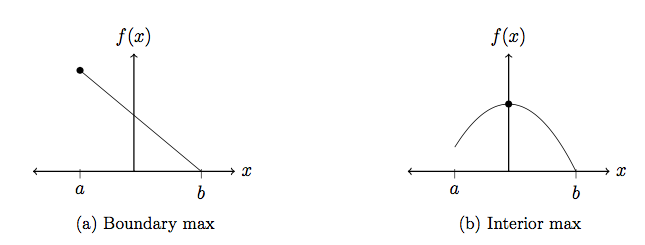
You can further evaluate this, but you get the point!

: What happens to the expression for Leibniz Rule when and are scalars and instead?

: Only the last term remains, actually this is how you will use if often.

3.3. Maxima and minima

3.3.1. Local maxima

* The point is a local maximum on the interval if for all
* If is an interior max of then is a *critical point* of .

At a critical point on the domain the second derivative can be used to check whether it is a maximum or a minimum.

1. Maximum
2. Minimum
3. Indeterminate

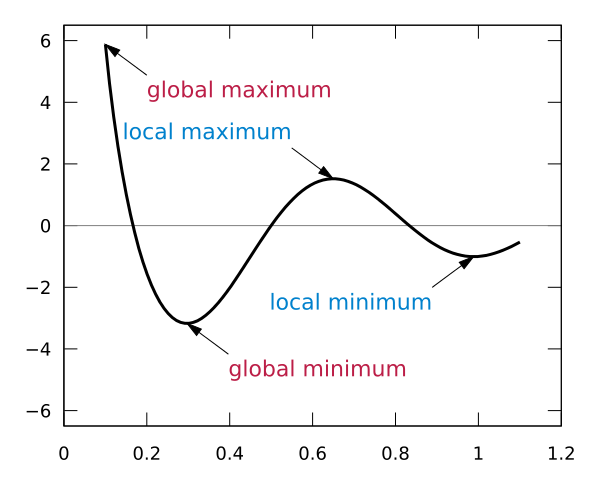
How do you remember these relations? This is how I do! (show)

3.3.2. Global maxima

The point is a global maximum if for all in the doman of Note that a global maximum need **not** be a *critical point*: The function has a global maximum when:

Theorem. Let be a twice differentiable function on the domain. If is a local maximum of and is the only critical point of on , then is a global maximum of on.

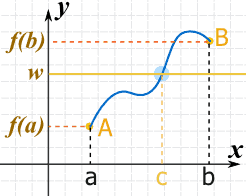
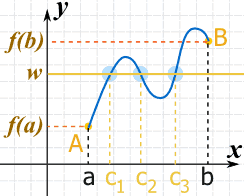
Let’s try an example. Which points are the local maxima and minima? (If they exist).



3.3.3 Other useful theorems:

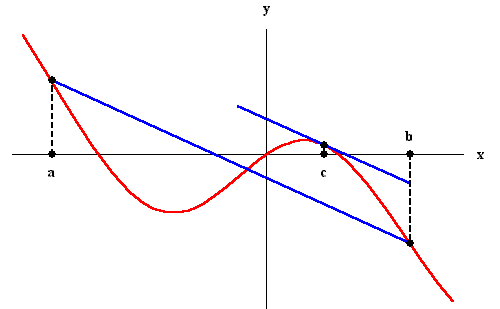
*Intermediate Value Theorem*. Let be a continuous function on the closed interval And let N be any given number between between and where. If . Then there exists a number in such that

Explanation: This is getting at the essence of not lifting your pencil when drawing a graph. w= N

 Can be more than 1 c: 

*Mean Value Theorem*: Suppose that is continuous on and differentiable on Then for some

Explanation: This is really the slope of the line passing through So the conclusion of the Mean Value Theorem states that there exists a point such that the tangent line is parallel to the line passing through



*Weierstrass’ Theorem*. A continuous function, on the closed and bounded interval attains both a local maximum and minimum.

*Taylor’s Theorem*. The Taylor series if the function f at a is given by:

Often, when you need to use Taylor series approximation, you will work with a manageable order of the series. For example, zeroth, the first, second, and third order of the Taylor polynomial are:

Let’s think about what is happening here. You are approximating and as you increase your order, you get closer and closer.

For example, is the same as the linear approximation of centered at , so it is often called “the first-order approximation of at (or near) .” is then called the quadratic, or second-order approximation, the cubic, or third-order approximation,

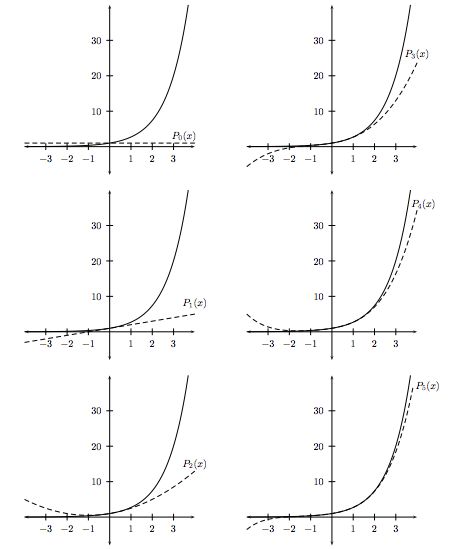
and so on.

*Example*. Find at x = 0 for the function .

Let’s center this at From the definition:

You can easily identify that:

So graphically, how good are we getting with the higher order?



Note that we used a nice example, but things are not always as neat. With varying functional forms and values of the approximation may not be as good as it sounds. Depending on what field courses you take you may or may not see a larger expose on Taylor series expansions. For now, this should be quite plenty.

3.4. Review of Euclidean n-space

A lot of what we do is in the n-space. If we are given , it can be viewed as a vector:

or

All your addition and subtraction operations need to be undertaken element by element. The *dot product* or *inner dorm* is given by:

3.5. Multi-variable calculus

Now we move to function of several variables.

You will work with utility functions where individuals get utility from multiple commodities, , with prices Or in production, you may have a production function where a particular output is made from several inputs. Consider a straightforward Cobb-Douglas production function using inputs and:

A function from to takes the numbers () in the domain of the function and assigns a number , i.e. Note that for each where ) we have .

A function from to takes the numbers () in the domain of the function and assigns a number , where , for all i.e.

Let’s consider a useful mapping that you will see often in Micro. You have m consumers, k commodities, and a utility mapping which is a function from to

3.6. Partial, total and higher order derivatives

3.6.1. Partial Derivatives

Partial Derivatives: A function on variables can be thought to have partial slopes, each giving only the rate at which would change if one , alone, where to change.

Let The partial derivatives of with respect to is defined as:

Here is an example:

*Example*. Given . Compute the partial derivative with respect to each element:

Taking partial derivative is not particularly harder than regular derivatives you have been taking. For example, if you are taking the derivative with respect to you can treat as a constant.

3.6.2. Total Derivatives

The total derivative of a function tells us how changes as we allow through to change *simultaneously*. The total differential at can be approximated as follows:

You may also encounter which is often called the *Jacobian derivative* of at . The Jacobian of is:

You may also see total derivatives used in the context of gradients. The gradient of at is the transpose of the derivative of itself:

It can be hard to picture the difference between some of these concepts. Mainly, the Jacobian and gradient vectors can be confusing. An example can help:

Suppose, , , then is a matrix given as:

Now suppose, , then the is the matrix:

Total derivative with chain rule:

Often, you will encounter total derivative in combination with the chain rule. Let’s go through simple examples of both and then I will pull an example from the Production mini.

*Example*. Let

Find the total derivative.

Now, you are given that and

Find

You can substitute all these and get your answer, but as in Micro you will be dealing with functions that are often unknown, let’s just denote that .

Then: . All these individual elements are easier to compute.

Plug in , and in terms of , and you have your answer in.

*Example*. When you are asked to solve problems using these math tools, rarely you will be directly told to compute a certain derivative or integral. You have to use your own judgement about what needs to be done and translate that in math.

Suppose , while in turn and How does change when changes? When changes? Let’s derive the expressions to answer both of these questions.

You may recall from an example earlier where we introduced outputs and inputs . You are given that , which is a standard function denoting the level of production. Remember that both and are vectors that that there are two inputs and one output . Hint, it is very simply, do not worry about chain rule here.

As a practice, totally differentiate with respect to .

3.6.3. Higher order derivatives

Think of one of the function’s partial derivatives, for example, the partial with respect to . We note that it is a function of variables as follows:

If we were to calculate the partial derivatives of we get the *gradient* of second order derivatives, which is:

A *Hessian Matrix* of the function contains all the possible second-order partial derivatives of the original function.

*Example*. Let

: What is the dimension of this Hessian for ?

: It is an matrix

Let’s find all that we need:

Young’s Theorem: Suppose that is twice continuously differentiable in . Then, for all and for each pair of indices :

Refer back to the example before, did we find that

Derive the Hessian for the function: . Can you check the condition from Young’s Theorem?

As , Young’s Theorem holds.

The Hessian,